General expressions can be utilized to determine velocity distributions for flow systems. This method is better than developing formulations peculiar to the specific problem at hand.

The general momentum equation is also called the equation of motion or the Navier-Stoke’s equation; in addition the equation of continuity is frequently used in conjunction with the momentum equation.
The equation of continuity is developed simply by applying the law of conservation of mass to a small volume element within a flowing fluid.

The momentum equation (the equation of motion or the Navier-Stoke’s equation) is an extension of previously written momentun balance.

Most of the fluid flow problems can be mathematically described by these two equations.
EQUATION OF CONTINUITY

Due to the conservation of mass;
(rate of mass accumulation) = (rate of mass in) – (rate of mass out)

Consider a stationary volume within a fluid moving with a velocity having the components $v_x$, $v_y$, and $v_z$. 
EQUATION OF CONTINUITY

The volume flow rate of fluid (VFR) in or out across the face = the velocity x the cross-sectional area

The rate of mass in or out through the face = (VFR) x density of fluid

\[ \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} = \Delta y \Delta z \left[ \rho v_x \bigg|_x - \rho v_x \bigg|_{x+\Delta x} \right] + \Delta x \Delta z \left[ \rho v_y \bigg|_y - \rho v_y \bigg|_{y+\Delta y} \right] + \Delta x \Delta y \left[ \rho v_z \bigg|_z - \rho v_z \bigg|_{z+\Delta z} \right] \]

Then dividing through by \( \Delta x \Delta y \Delta z \), and taking the limit as these dimensions approach zero, we get the equation of continuity:

\[ \frac{\partial \rho}{\partial t} = -\left[ \frac{\partial}{\partial x} \rho v_x + \frac{\partial}{\partial y} \rho v_y + \frac{\partial}{\partial z} \rho v_z \right] \]
EQUATION OF CONTINUITY

If the fluid density is constant then the continuity equation reduces to

\[
0 = \left[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]
\]

or in vector notation \( \nabla \cdot \mathbf{v} = 0 \)
Table 2.1 The continuity equation in different coordinates systems

Rectangular coordinates $(x, y, z)$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \quad (A)$$

Cylindrical coordinates $(r, \theta, z)$:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(\rho rv_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho rv_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \quad (B)$$

Spherical coordinates $(r, \theta, \phi)$:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(\rho rv_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\rho rv_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}(\rho v_\phi) = 0 \quad (C)$$

THE MOMENTUM EQUATION

When the previous momentum balance equation is extended to include unsteady-state systems;
(rate of momentum accumulation) = (rate of momentum in) – (rate of momentum out) + (sum of forces acting on the system)

Fig. 2.5 Momentum transport (x-component) due to viscosity into the volume element. (a) Directions of viscous momentum transport. (b) Directions of stresses.
The x- component of the momentum equation

\[
\frac{\partial}{\partial t} \rho v_x = - \left[ \frac{\partial}{\partial x} \rho v_x v_x + \frac{\partial}{\partial y} \rho v_y v_x + \frac{\partial}{\partial z} \rho v_z v_x \right]
\]

\[
- \left[ \frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right] \frac{\partial P}{\partial x} + \rho g_x
\]

To describe the general case, all three components (x, y and z) are needed.
Table 2.2 The momentum equation in rectangular coordinates \((x, y, z)\)

In terms of \(\tau\):

**x-component**
\[
\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} - \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) + \rho g_x \tag{A}
\]

**y-component**
\[
\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} - \left( \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) + \rho g_y \tag{B}
\]

**z-component**
\[
\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} - \left( \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \rho g_z \tag{C}
\]

In terms of velocity gradients for a Newtonian fluid with constant \(\rho\) and \(\eta\):

**x-component**
\[
\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \eta \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_x \tag{D}
\]

**y-component**
\[
\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \eta \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_y \tag{E}
\]

**z-component**
\[
\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \eta \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \tag{F}
\]